DEFORMATION AND FRACTURE OF CRYSTALLINE MATERIALS UNDER COMPLEX LOADING-PROGRAM CONDITIONS

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In predicting the deformation and conditions of fracture of materials under complex loading program conditions one must consider the entire duration of the deformation process.

The problem becomes complicated in cases of a simultaneous operation of various mechanisms of deformation and fracture, e.g., when plastic deformation is superposed (once or repeatedly) on creep.

A promising phenomenological approach to this problem may be based on concepts of the mechanical equation of state of materials. A hypothesis of the existence of the equation of state depending on a finite number of structural parameters was formulated by Kröner [1] for the case of the three-dimensional law of plasticity and by Rabotnov [2] for the case of creep and fracture under uniaxial stress state conditions.

This article is concerned with the application of the hypothesis of the mechanical equation of state to the problem of deformation and fracture of materials (in the uniaxial case) under complex loading program conditions.

1. Basic assumptions. The material is loaded in accordance with an arbitrary program determined for instance by the time dependence of stress $\sigma(\tau)$ and $t(\tau)$.

To formulate a phenomenological description of the behavior of the material under these conditions, we make the following assumptions.

(a) The strain components are additive

$$d\varepsilon = d\vartheta + de + dp, \quad \vartheta = \sigma / E + \varkappa t,$$
$$E = E(t), \quad \varkappa = \varkappa(t).$$

Here ε, ϑ, e, and p denote the total strain, the reversible strain (elastic and thermal), the plastic strain, and the creep strain, respectively, E denoting the elasticity modulus.

(b) The variation in the plastic strain on the differential portion of the loading path is described by

$$de = R_1 d\tau + R_2 d\sigma + R_3 dt. \tag{1.1}$$

The terms in the right part of (1.1) describe the simultaneous influence of time-controlled processes (diffusion, aging) on the resistance to plastic deformation ($R_1 d\tau$), the momentary isothermal variation in the plastic strain due to stress variation ($R_2 d\sigma$), and the influence of temperature changes on the resistance to plastic deformation ($R_3 dt$). The creep strain is given by

$$dp = R_4 d\tau \tag{1.2}$$

where R_4 is the creep rate.

(c) Values R_l ($l = 1, 2, 3, 4$) are functions of time τ, stress σ, temperature t and structural parameters q_r ($r = 1, 2, 3, \ldots, s$)

$$R_l = R_l(\tau, \sigma, t, q_1, q_2, \ldots, q_s). \tag{1.3}$$

The structural parameters are described [2] by relations of the following type:

$$dq_i = a_{i1} d\tau + a_{i2} d\sigma + a_{i3} dt, \tag{1.4}$$

$$a_{ij} = a_{ij}(\tau, \sigma, t, q_1, q_2, \ldots, q_s). \tag{1.5}$$

Each of the coefficients R_l and a_{ij} has two branches: two values corresponding to any given combination of τ, σ, t and q_r.

One of these branches (R'_l, a'_{ij}) corresponds to active plastic deformation, while the other (R''_l, a''_{ij}) relates to the unloading stage.

(d) The moment of the start of unloading coincides with the maximum $|e|$. The condition for this maximum may be written in the form

$$e(R_1' + R_2' d\sigma/d\tau + R_3' dt/d\tau) \leqslant 0. \tag{1.6}$$

Here e denotes the plastic strain at the beginning of the unloading stage.

If the condition for the maximum $|e|$ is expressed through coefficients R''_l, we obtain the necessary and sufficient condition for unloading in the form

$$R_1'' + R_2'' d\sigma/d\tau + R_3'' dt/d\tau = 0. \tag{1.7}$$

Conditions (1.6) can be satisfied in many ways, i.e., $d\sigma/d\tau$ and $dt/d\tau$ during unloading are not unique values. Hence it follows that (1.7) should be identically satisfied, i.e., at the initial unloading moment $R''_l = 0$ ($l = 1, 2, 3$).

Functions (1.3) and (1.5) are unknown. A possible way of solving the problem in question would entail selecting a hypothetical form of these functions on the basis of a heuristic generalization of simple experiments and subsequently verifying the conclusions reached by experiments under complex loading conditions.

The variation in the mechanical properties of solids is due to both time-controlled processes (aging, diffusion, etc.) and irreversible deformations (plastic and creep strains).

Bearing in mind the direct influence of irreversible strains on the variation in mechanical properties, it is convenient to write (1.4) in the following equivalent form

$$dq_i = a_{i1} d\tau + a_{i2} de + a_{i3} dp + a_{i4} dt. \tag{1.8}$$

The previous symbols a_{ij} are used as the new coefficients in (1.8).

Let us apply the above concepts to certain problems of deformation and fracture of materials under complex loads at a constant temperature.

As the structural parameters, let us take $q_1 = u$, $q_2 = v$, $q_3 = \psi$, where

$$u = \int \sigma de, \qquad v = \int \sigma dp. \tag{1.9}$$

Here the integration extends over the entire duration of the deformation process. The parameter u represents the specific energy dissipated by the materials as a result of short-lasting plastic deformation, while v is the energy dissipated as a result of creep. The third parameter ψ represents the degree of damage of the material which is equal to zero in the initial state and to one at the moment of fracture.

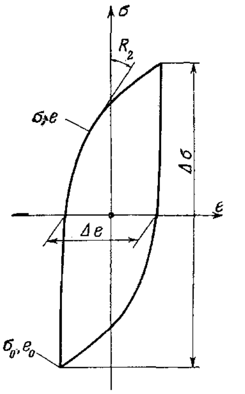

Fig. 1

The introduction of parameters u and v reflects the fact that structural changes and the character of fracture due to short-lasting plastic deformation and due to creep are different, and this difference can be taken into account by introducing not less than two structural parameters.

If the loading program $\sigma(\tau)$ and the starting conditions are known, the system of equations (1.1), (1.2), and (1.9) is not sufficient to determine all the strength characteristics; an additional relation of the (1.8) type is necessary in the form

$$d\psi = h_1 du + h_2 dv. \tag{1.10}$$

Condition (1.10) represents an assumption that the damage of the material is produced by irreversible strains. Values h_1 and h_2 are obviously functions of τ, σ, u, v, and ψ. Taking this into account and using (1.1), (1.2), (1.9), and (1.10), one can write the system of equations determining the behavior of a nonaging material in the form

$$de = R_2 d\sigma, \quad dp = R_4 d\tau, \quad du = \sigma de, \quad dv = \sigma dp\,, \tag{1.11}$$

$$d\psi = h_1 du + h_2 dv. \tag{1.12}$$

Here $1/R_2$ is the plasticity modulus.

2. **Fatigue.** In this case it is necessary to take in the system of equations (1.11) and (1.12) $R_4 \equiv 0$ and to choose the form of functions R_2 and h_1. The function R_2 can be obtained from the law of repeated deformation

$$d(e - e_0) = m\left(\frac{\sigma - \sigma_0}{2\sigma_k}\right)^{m-1} \frac{d(\sigma - \sigma_0)}{E}.$$

Here σ_0 and e_0 denote, respectively, the stress and plastic strain at the moment of stress reversal (Fig.1); m is a structure-insensitive characteristic of the material; and σ_k is a structure-sensitive parameter analogous to the "instantaneous" yield point.

Relation (2.1) expresses the Mazing principle [3] with the parameter σ_k introduced to describe the instability of the plastic hysteresis loop during cyclic loading. From the standpoint of accepted phenomenological positions, one should regard σ_k as a new structural parameter and add a relation

$$d\sigma_k = g_1 du + g_2 d\psi. \tag{2.2}$$

In a simplified variant let us assume integrability of (2.2) and the existence of a relation in the form

$$\sigma_k = \sigma_k(u, \psi). \tag{2.3}$$

The hypothesis (2.3) can be used to interpret data on the variation in the plastic hysteresis loop during cyclic deformation [4] and to describe the differences in the behavior of materials that harden, are stable, or weaken under the influence of cyclic loads. It appears that the most typical variation in σ_k during cyclic loading is when the influence of strain hardening predominates in the initial stages leading to a slight increase in σ_k, after which σ_k begins to decrease with increasing number of stress cycles as a result of weakening measured in terms of the degree of damage ψ.

Certain experimental results point to the existence of a stabilization stage to which corresponds a constant (or almost constant) value of σ_k. Let us write (2.3) in the form

$$\sigma_k = \sigma_{k0}(1 - \psi) S(u). \tag{2.4}$$

Here S(u) is the hardening function reflecting the influence of strain-hardening on the yield point and σ_{k0} is a material constant.

Let us choose the function $h_1(\sigma, u, \psi)$ in the form

$$h_1 = c_1 \left(\frac{\sigma}{1 - \psi}\right)^{\gamma} F_1(u). \tag{2.5}$$

Here c_1 and γ are material constants.

The hypothesis (2.5) is based on regarding ψ as the degree of weakening of the material cross section [5]. The function $F_1(u)$ should take into account the effect of strain hardening on the material strength.

Let us consider cyclic deformation of a material using relations (2.1), (2.4) and (2.5). The change in u and ψ during one deformation cycle is found from relations (1.11), (2.1), (1.12) and (2.5), bearing in mind that the variation in u and ψ during one cycle is small and that these values in the right sides of (2.4) and (2.5) may be taken as constant. In this way we obtain

$$\frac{du}{dn} = \xi_1 \qquad \left(\xi_1 = \oint \sigma de\right), \tag{2.6}$$

$$\frac{d\psi}{dn} = c_1 \frac{F_1(u)}{(1 - \psi)^{\gamma}} \zeta_1 \qquad \left(\zeta_1 = \oint |\sigma|^{\gamma} \sigma de\right). \tag{2.7}$$

Here the integration extends over one cycle, and n denotes the number of cycles.

Using (2.1), from (2.6) and (2.7) we obtain for a symmetrical cycle

$$\xi_1 = \frac{I(m, 0)}{E(2\sigma_k)^{m-1}} \Delta\sigma^{m+1},$$

$$\zeta_1 = \frac{I(m, \gamma)}{E(2\sigma_k)^{m-1}} \Delta\sigma^{\gamma+m+1}$$

$$\left(I(m, \gamma) = \frac{m}{2^{\gamma+m}} \int_{-1}^{+1} |x|^{\gamma} (1+x)^{m-1} x dx\right). \qquad (2.8)$$

Here $\Delta\sigma$ is the stress interval (Fig. 1). From (2.6) and (2.7) and taking into account (2.8), we obtain

$$d\psi / du = c_2 \Delta\sigma^{\gamma} F_1(u) / (1-\psi)^{\gamma}, \qquad (2.9)$$

where c_2 is a material constant.

a) Fatigue at $\Delta\sigma$ = const. Taking for $F_1(u)$ a power function

$$F_1(u) = u^{-\delta} \qquad (\delta = \text{const}) \qquad (2.10)$$

from (2.9) at $\Delta\sigma$ = const (high-endurance fatigue) we find

$$1 - (1-\psi)^{\gamma+1} = \frac{\gamma+1}{1-\delta} c_2 \Delta\sigma^{\gamma} u^{1-\delta}. \qquad (2.11)$$

At fracture ψ = 1 and u = U, so that from (2.11) we have

$$\Delta\sigma = H U^{-(1-\delta)/\gamma}. \qquad (2.12)$$

Here H is a material constant and U is the value of u at the moment of fracture.

If it is assumed that there exists a stage of loop stabilization, one may take σ_k =const; then, from (2.6), the first relation of (2.8), and from (2.12), we obtain

$$u \sim \Delta\sigma^{m+1} n, \quad U \sim \Delta\sigma^{m+1}, \; N, \; U \sim N^{\mu}$$
$$\left(\mu = \frac{\gamma}{\gamma + (m+1)(1-\delta)}\right). \qquad (2.13)$$

The symbol ~ denotes proportionality, and N is the number of cycles to fracture.

The error due to assuming σ_k = const is compensated by the fact that function $F_1(u)$ reflecting the real instability of the material is determined from fatigue test results.

Using (2.12) and (2.13), one can obtain the equation for fatigue in its usual form:

$$N \Delta\sigma^b = K \qquad \left(b = m + 1 + \frac{\gamma}{1-\delta}\right). \qquad (2.14)$$

Here K and b are constants determined by fatigue tests at $\Delta\sigma$ = const.

Relation (2.9) can be used to compute the conditions of fracture at variable $\Delta\sigma$.

Let us consider the case of stepwise loading, when the material is subjected to n_1 loading cycles at $\Delta\sigma_1$, $(n_2 - n_1)$ cycles at $\Delta\sigma_2$, etc.

Integrating (2.9) and taking into account (2.12), we obtain the basic condition for fracture under stepwise loading conditions in the form

$$\sum_{i=1}^{s} \frac{u_i^{1-\delta} - u_{i-1}^{1-\delta}}{U_i^{1-\delta}} = 1 \qquad (u_0 = 0). \qquad (2.15)$$

Here u_i is the value of u at the end of the i-th cyclic deformation interval.

The relationship between U_i and $\Delta\sigma_i$ is described by (2.12). At σ_k = const, it follows from (2.6) that

$$u_i - u_{i-1} \sim \Delta\sigma_i^{m+1} (n_i - n_{i-1}). \qquad (2.16)$$

Here n_i denotes the number of loading cycles accumulated at the end of the i-th interval. Taking into account (2.12) and (2.13), from (2.16) one can obtain

$$\frac{u_i}{U_i} = \frac{u_{i-1}}{U_{i-1}} \left(\frac{N_{i-1}}{N_i}\right)^{\mu} + \frac{n_i - n_{i-1}}{N_i},$$

$$\frac{u_{i-1}}{U_i} = \frac{u_{i-1}}{U_{i-1}} \left(\frac{N_{i-1}}{N_i}\right)^{\mu}, \qquad (2.17)$$

which makes it possible to express condition (2.15) as a function of n_i and N_i. Here N_i is the number of cycles to fracture at $\Delta\sigma_i$ = const.

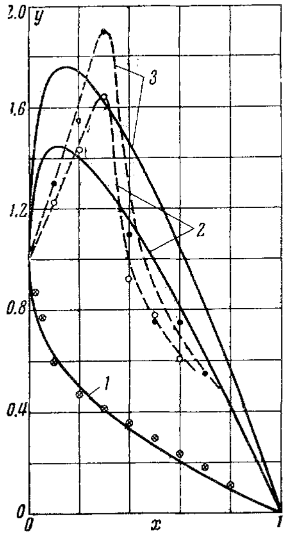

Fig. 2

The conditions for fracture for a one-step loading program are obtained from (2.15) and (2.17) in the form

$$(1 - a^{\mu(1-\delta)}) x^{1-\delta} + (y + a^{\mu} x)^{1-\delta} = 1$$
$$\left(x = \frac{n_1}{N_1}, \quad y = \frac{n_2 - n_1}{N_2}, \quad a = \frac{N_1}{N_2}\right). \qquad (2.18)$$

For δ = 0 we obtain from (2.18) the rule of linear additivity of fatigue damage: x + y = 1. For values $0 < \delta < 1$, relation (2.18) predicts an increase in endurance in the initial stage at $a > 1$ (i.e., when the lower stress is applied first), which corresponds to the known prestraining effect. At $a < 1$ (i.e., when the higher stress is applied first), relation (2.18) predicts a sharp reduction in the endurance in the initial fatigue stages. Both these facts are observed in a majority of tests under stepwise cyclic loading conditions. A comparison between experimental data from [6] and relation (2.18) is shown in Fig. 2.

Curve 1 relates to a stepwise variation in the stress stress amplitude from σ_1 = 24.3 ($N_1 = 1.7 \cdot 10^5$) to σ_2 = = 19.8 ($N_2 = 2.25 \cdot 10^6$) curves 2 correspond to σ_1 = 19.8 ($N_1 = 2.25 \cdot 10^6$) σ_2 = 24.3 ($N_2 = 1.7 \cdot 10^5$); and curves 3 to σ_1 = 19.8 ($N_1 = 2.25 \cdot 10^6$), σ_2 = 26.5 ($N_2 = 7 \cdot 10^4$). (Stresses are given in kgf/mm^2.)

The constant μ in (2.18) is taken equal to 0.5 (on the basis of considerations outlined below). Constant δ = 0.593 was calculated from the condition of congruence of relation (2.18) with experiment at a_1 = = 0.0755; these experimental results are shown in Fig. 2 by crosses.

Solid curves 2 and 3 were plotted from (2.18) for $\delta = 0.593$, while dashed curves 2 and 3 represent the corresponding experimental data for $a_2 = 13.24$ and $a_3 = 32.2$.

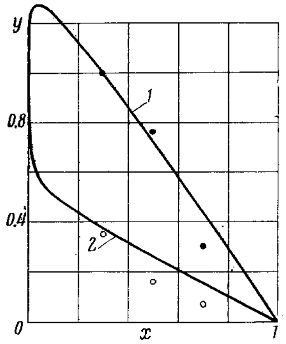

Fig. 3

b) Fatigue at Δe = const. The relation between Δe and $\Delta\sigma$ is found from (2.1) in the form

$$\Delta e = \left(1 - \frac{1}{2^{m-1}}\right)\frac{\Delta\sigma}{E\,(2\sigma_k)^{m-1}}.$$

Using this relation and (2.4), we find a solution of Eq. (2.9) in the form

$$1 - (1 - \psi)^{1+\gamma/m} = c_3 \Delta e^{\gamma/m} \int_0^u F(z)\,dz\,,$$

$$F(z) = F_1(z)\,S(z)^{\gamma(1-1/m)}. \qquad (2.19)$$

Here c_3 is a material constant.

If one starts from (2.10), solution (2.19) for σ_k = = const is analogous to solution (2.11). For this reason all the derivative relations have the same form as the relations for the case of $\Delta\sigma$ = const. The law of fracture at Δe = const is obtained from (2.19) in the form of the Koffin formula

$$N\Delta e^k = C \qquad \left(k = \frac{m+1}{m} + \frac{\gamma}{m(1-\delta)}\right). \qquad (2.20)$$

Here C and k are material constants determined from experimental data on low-endurance fatigue. In many cases $k \approx 2$.

The condition for fracture in the case of step-wise variation in Δe is given by the same formulas ((2.15), (2.17), (2.18)) as in the case of stepwise variation in $\Delta\sigma$.

From (2.13) and (2.20) it follows that μ and k are related as

$$\mu = (k-1)/k - 1/mk.$$

From this it will be seen that $\mu \approx 0.5$ in all cases in which $k \approx 2$, since 1/mk is small (0.02–0.05). Formulas (2.15) and (2.18) contain another unknown constant δ which must be determined by experiment under nonsteady-state conditions (e.g., by tests in which Δe is varied stepwise from one level to another).

In Fig. 3 experimental data on stepwise-cyclic loading [8] are shown side by side with graphs of formula (2.18) plotted for $\mu = 0.5$ and $\delta = 0.9$, curves 1 and 2 corresponding to $a_1 = 2.25$ and $a_2 = 0.231$, respectively; constant δ was determined from the condition that the theoretical curve (2.18) should pass through one experimental point ($x = 0.3$, $y = 1$).

3. Creep. Let us take $R_2 \equiv 0$ in (1.11), (1.12) and apply this system of equations to determine the conditions of fracture as a result of creep at a variable stress.

The parameter v will be described by a relation slightly different from the fourth equation in (1.11),

$$dv = \sigma^{\alpha+1} d\tau\,, \qquad (3.1)$$

where α is a constant in the expression for steady-state creep

$$dp = D\sigma^\alpha d\tau.$$

Using the equation of strength in the form

$$d\psi = M\left(\frac{\sigma}{1-\psi}\right)^{\gamma'} F_2(v)\,dv \qquad (F_2(v) = v^{-\delta'})$$

we obtain a complete (formal) analogy of the equations of strength in fatigue and creep. As a result, the condition for fracture in stepwise creep at stresses σ_1, σ_2, ... σ_s has a form analogous to (2.15),

$$\sum_{i=1}^{s} \frac{v_i^{1-\delta'} - v_{i-1}^{1-\delta'}}{V_i^{1-\delta'}} = 1 \qquad (v_0 = 0)\,,$$

where V_i is the limiting value of v_i corresponding to fracture at a constant stress σ_i. At the same time, in analogy to (2.12), we have

$$\sigma_i = B' V_i^{-(1-\delta')/\gamma'}.$$

From (3.1) it follows that at a constant σ

$$v = \sigma^{\alpha+1}\tau, \qquad V = \sigma^{\alpha+1}T\,,$$

where T denotes the time to fracture at a stress σ, and τ is the time during which the material is acted on by the stress σ. For stepwise loading, the final formulas describing the conditions for fracture through τ_i and T_i are (due to the above mentioned analogy) analogous to the corresponding formulas for fatigue.

To obtain the criteria of fracture in creep, the following substitutions must be carried out in formulas (2.15), (2.17) and (2.18); v must be substituted for u_i, V_i for U_i, τ_i for n_i, T_i for N_i, δ' for δ, and μ' for μ. At the same time

$$\mu' = 1 - (\alpha + 1)/\beta\,,$$

where β is a constant in the expression for long-term strength:

$$T\sigma^\beta = B.$$

Relations obtained in this way make it possible to describe the nonlinear accumulation of damage observed in several investigations (e.g., [9]).

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